

**CALCULUS 1E****ASSIGNMENT 1****PART 1**

a) Prove using mathematical induction that:

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

For  $n = 1$

$$1^2 = \frac{1}{6}(1+1)(2+1)$$

$$1 = \frac{1}{6} \times 2 \times 3$$

$$1 = 1$$

For  $n = n + 1$

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 + (n+1)^2 = \frac{1}{6}n(n+1)(n+2)(2n+3)$$

$$LHS = \frac{1}{6}n(n+1)(2n+1) + (n+1)^2$$

$$\frac{1}{6}(n+1)[n(2n+1) + 6(n+1)]$$

$$\frac{1}{6}(n+1)(2n^2 + n + 6n + 6)$$

$$\frac{1}{6}(n+1)(2n^2 + 7n + 6)$$

$$\frac{1}{6}(n+1)(n+2)(2n+3) = RHS \quad \text{QED}$$

Use the formula above to:

(I) Find the value of  $1^2 + 2^2 + 3^2 + \dots + 10^2$

$$\sum_{i=1}^{10} i^2 = 1^2 + 2^2 + 3^2 + \dots + 10^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\frac{1}{6} \times 10(10+1)(20+1)$$

$$\frac{1}{6} \times 10 \times 11 \times 21$$

= 385

(ii) Find an expression for  $\left(\frac{a}{4}\right)^2 + \left(\frac{2a}{4}\right)^2 + \left(\frac{3a}{4}\right)^2 + \dots + \left(\frac{ra}{4}\right)^2$

$$\sum_{i=1}^r \left[ i \left( \frac{a}{4} \right) \right]^2 = \left( \frac{a}{4} \right)^2 + \left( \frac{2a}{4} \right)^2 + \left( \frac{3a}{4} \right)^2 + \dots + \left( \frac{ra}{4} \right)^2 = \left( \frac{a}{4} \right)^2 \left[ \frac{1}{6} r(r+1)(2r+1) \right]$$

b) Develop a formula for the following

$$\sum_{i=1}^n (2i-1)^3 = 1^3 + 3^3 + 5^3 + \dots + (2n-1)^3$$

n	1	2	3	4	5	6	7	8	9	10
	1 <sup>3</sup>	3 <sup>3</sup>	5 <sup>3</sup>	7 <sup>3</sup>	9 <sup>3</sup>	11 <sup>3</sup>	13 <sup>3</sup>	15 <sup>3</sup>	17 <sup>3</sup>	19 <sup>3</sup>
	1	28	153	496	1225	2556	4753	8128	13041	19900
	1 <sup>2</sup> × 1	2 <sup>2</sup> × 7	3 <sup>2</sup> × 17	4 <sup>2</sup> × 31	5 <sup>2</sup> × 49	6 <sup>2</sup> × 71				

= n<sup>2</sup>(2n<sup>2</sup> - 1)

**Proof by mathematical induction**

$$\sum_{i=1}^n (2i-1)^3 = 1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = n^2(2n^2 - 1)$$

For n = 1

$$1^3 = 1^2(2 \times 1^2 - 1)$$

$$1 = 2 - 1$$

$$1 = 1$$

For n = n + 1

$$\sum_{i=1}^n (2i-1)^3 = 1^3 + 3^3 + 5^3 + \dots + [2(n+1)-1]^3 = n^2(2n^2 - 1)$$

$$(2n+2-1)^3$$

$$(2n+1)^3$$

$$\sum_{i=1}^n (2i-1)^3 = 1^3 + 3^3 + 5^3 + \dots + (2n+1)^3 = (n+1)^2 [2(n+1)^2 - 1]$$

$$(n^2 + 2n + 1)(2n^2 + 4n + 2 - 1)$$

$$\begin{aligned}
 & (n^2 + 2n + 1)(2n^2 + 4n + 1) \\
 & 2n^4 + 4n^3 + n^2 + 4n^3 + 8n^2 + 2n + 2n^2 + 4n + 1 \\
 & RHS = 2n^4 + 8n^3 + 11n^2 + 6n + 1 \\
 & \sum_{i=1}^n (2i-1)^3 = 1^3 + 3^3 + 5^3 + \dots + (2n+1)^3 = 2n^4 + 8n^3 + 11n^2 + 6n + 1
 \end{aligned}$$

$$\begin{aligned}
 LHS &= n^2(2n^2 - 1) + (2n + 1)^3 \\
 & 2n^4 - n^2 + (2n + 1)(2n + 1)(2n + 1) \\
 & 2n^4 - n^2 + (4n^2 + 2n + 2n + 1)(2n + 1) \\
 & 2n^4 - n^2 + (4n^2 + 4n + 1)(2n + 1) \\
 & 2n^4 - n^2 + 8n^3 + 4n^2 + 8n^2 + 4n + 1 + 2n \\
 & 2n^4 - n^2 + 8n^3 + 12n^2 + 6n + 1 \\
 & 2n^4 + 8n^3 + 11n^2 + 6n + 1 = RHS \quad QED
 \end{aligned}$$

## PART 2

### Cylinders inside cone

$$\begin{aligned}
 V_{3 \text{ cylinders}} &= \mathbf{P} \left( \frac{h}{4} \right) \left( \frac{r}{4} \right)^2 + \mathbf{P} \left( \frac{h}{4} \right) \left( \frac{r}{2} \right)^2 + \mathbf{P} \left( \frac{h}{4} \right) \left( \frac{3r}{4} \right)^2 \\
 & \mathbf{P} \frac{h}{4} \left[ \left( \frac{r}{4} \right)^2 + \left( \frac{r}{2} \right)^2 + \left( \frac{3r}{4} \right)^2 \right] \\
 & \mathbf{P} \frac{h}{4} \left( \frac{r^2}{16} + \frac{r^2}{4} + \frac{9r^2}{16} \right) \\
 & \mathbf{P} \frac{h}{4} r^2 \left( \frac{1}{16} + \frac{1}{4} + \frac{9}{16} \right) \\
 & \mathbf{P} \left( \frac{h}{4} \right) r^2 \times \frac{7}{8} \\
 & \mathbf{P} \left( \frac{7h}{32} \right) r^2 \\
 & \mathbf{P} \frac{7}{32} hr^2 = \frac{7}{32} \mathbf{P} hr^2
 \end{aligned}$$

The volume of a cone is  $\frac{1}{3} \mathbf{P} r^2 h$

$$\frac{1}{3} \mathbf{P} r^2 h > \frac{7}{32} \mathbf{P} r^2 h$$

## Cylinders outside cone

$$\begin{aligned}
 V_{4 \text{ cylinders}} &= \mathbf{p} \left( \frac{h}{4} \right) \left( \frac{r}{4} \right)^2 + \mathbf{p} \left( \frac{h}{4} \right) \left( \frac{r}{2} \right)^2 + \mathbf{p} \left( \frac{h}{4} \right) \left( \frac{3r}{4} \right)^2 + \mathbf{p} \left( \frac{h}{4} \right) r^2 \\
 &= \mathbf{p} \frac{h}{4} \left( \frac{r^2}{16} + \frac{r^2}{4} + \frac{9r^2}{16} + r^2 \right) \\
 &= \mathbf{p} \left( \frac{h}{4} \right) r^2 \left( \frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1 \right) \\
 &= \mathbf{p} \left( \frac{h}{4} \right) r^2 \times 1 \frac{7}{8} \\
 &= \mathbf{p} \left( \frac{h}{4} \right) r^2 \times \frac{15}{8} \\
 &= \mathbf{p} \left( \frac{15h}{32} \right) r^2 \\
 &= \frac{15}{32} \mathbf{p} h r^2
 \end{aligned}$$

The volume of a cone is  $\frac{1}{3} \mathbf{p} r^2 h$

$$\frac{1}{3} \mathbf{p} r^2 h < \frac{15}{32} \mathbf{p} h r^2$$

Investigate how accurate these rules are for cylinders of particular sizes

## Cylinders inside cones

### Cone of height 1 and radius 2

$$\begin{aligned}
 V_{\text{approx}} &= \frac{7}{32} \mathbf{p} \times 1 \times 2^2 \\
 &= \frac{7}{32} \mathbf{p}^4 \\
 &= 2.7489
 \end{aligned}$$

$$\begin{aligned}
 V_{\text{actual}} &= \frac{1}{3} \mathbf{p} \times 1 \times 2^2 \\
 &= \frac{1}{3} \mathbf{p}^4 \\
 &= 4.189
 \end{aligned}$$

### Cone of height 5 and radius 15

$$V_{approx} = \frac{7}{32} \mathbf{p} \times 5 \times 15^2$$

$$\frac{7}{32} \mathbf{p} \times 5 \times 225$$

$$= 773.13$$

$$V_{actual} = \frac{1}{3} \mathbf{p} \times 5 \times 15^2$$

$$\frac{1}{3} \mathbf{p} \times 5 \times 225$$

$$= 1178.1$$

Cone of height 10 and radius of 20

$$V_{approx} = \frac{7}{32} \mathbf{p} \times 10 \times 20^2$$

$$\frac{7}{32} \mathbf{p} \times 10 \times 400$$

$$= 2748.9$$

$$V_{actual} = \frac{1}{3} \mathbf{p} \times 10 \times 20^2$$

$$\frac{1}{3} \mathbf{p} \times 10 \times 400$$

$$= 4188.8$$

Cylinders outside cones

Cone of height 1 and radius 2

$$V_{approx} = \frac{15}{32} \mathbf{p} \times 1 \times 2^2$$

$$\frac{15}{32} \mathbf{p} \times 4$$

$$= 5.89$$

$$V_{actual} = 4.188$$

Cone of height 5 and radius 15

$$V_{approx} = \frac{15}{32} \mathbf{p} \times 5 \times 15^2$$

$$\frac{15}{32} \mathbf{p} \times 5 \times 225$$

$$= 1656.7$$

$$V_{actual} = 1178.1$$

Cone of height 10 and radius 20

$$V_{approx} = \frac{15}{32} \mathbf{p} \times 10 \times 20^2$$

$$\frac{15}{32} \mathbf{p} \times 10 \times 400$$

$$V_{actual} = 4188.8$$

$$= 5890.5$$

These rules aren't very accurate for any size of cone.

By increasing the number of cylinders investigate how the formula for a cone may be discovered,

Formula for the volume of n number of cylinders inside a cone.

$$\begin{aligned}
 V_{3\text{cylinders}} &= \pi \left( \frac{h}{4} \right) \left( \frac{r}{4} \right)^2 + \pi \left( \frac{h}{4} \right) \left( \frac{2r}{4} \right)^2 + \pi \left( \frac{h}{4} \right) \left( \frac{3r}{4} \right)^2 \\
 &= \pi \frac{h}{4} \left[ \left( \frac{r}{4} \right)^2 + \left( \frac{2r}{4} \right)^2 + \left( \frac{3r}{4} \right)^2 \right] \\
 &= \pi \frac{h}{4} \left( \frac{r^2}{16} + \frac{4r^2}{16} + \frac{9r^2}{16} \right) \\
 &= \pi \frac{h}{4} \left( \frac{1}{16} + \frac{4}{16} + \frac{9}{16} \right) \\
 &= \frac{1}{n+1} \left[ \frac{\frac{1}{6} n(n+1)(2n+1)}{(n+1)^2} \right] \pi h r^2 \\
 &= \frac{1}{n+1} \left[ \frac{\frac{1}{6} n(2n+1)}{n+1} \right] \pi h r^2 \\
 &= \frac{\frac{1}{6} n(2n+1)}{(n+1)^2} \pi h r^2
 \end{aligned}$$

Formula for the volume of n number of cylinders outside a cone.

$$\begin{aligned}
 V_{4\text{cylinders}} &= \pi \left( \frac{h}{4} \right) \left( \frac{r}{4} \right)^2 + \pi \left( \frac{h}{4} \right) \left( \frac{2r}{4} \right)^2 + \pi \left( \frac{h}{4} \right) \left( \frac{3r}{4} \right)^2 + \pi \left( \frac{h}{4} \right) \left( \frac{4r}{4} \right)^2 \\
 &= \pi \frac{h}{4} \left[ \left( \frac{r}{4} \right)^2 + \left( \frac{2r}{4} \right)^2 + \left( \frac{3r}{4} \right)^2 + \left( \frac{4r}{4} \right)^2 \right]
 \end{aligned}$$

$$\begin{aligned}
 & \mathbf{p} \frac{h}{4} \left( \frac{r^2}{16} + \frac{4r^2}{16} + \frac{9r^2}{16} + \frac{16r^2}{16} \right) \\
 & \frac{1}{4} \left( \frac{1}{16} + \frac{4}{16} + \frac{9}{16} + \frac{16}{16} \right) \mathbf{p} h r^2 \\
 & \frac{1}{n} \left[ \frac{\frac{1}{6} n(n+1)(2n+1)}{n^2} \right] \mathbf{p} h r^2 \\
 & \frac{1}{n} \left[ \frac{\frac{1}{6} (n+1)(2n+1)}{n} \right] \mathbf{p} h r^2 \\
 & \frac{1}{6} \frac{(n+1)(2n+1)}{n^2} \mathbf{p} h r^2
 \end{aligned}$$

As the value for  $n$  tends to infinity the value for  $\frac{1}{6} \frac{(n+1)(2n+1)}{n^2}$  gets closer and closer to  $\frac{1}{3}$  but will always be greater than  $\frac{1}{3}$ .

As the value for  $n$  tends to infinity the value for  $\frac{1}{6} \frac{n(2n+1)}{(n+1)^2}$  gets closer and closer to  $\frac{1}{3}$  but will always be less than  $\frac{1}{3}$ .

By making  $n$  a very high number like 1000000 and getting the values for  $\frac{1}{6} \frac{(n+1)(2n+1)}{n^2}$  and  $\frac{1}{6} \frac{n(2n+1)}{(n+1)^2}$  we see that these are both very close to  $\frac{1}{3}$ , by getting the average of these two numbers we can get an almost exact value and therefor hypothesize that the formula for the volume of a cone is indeed  $\frac{1}{3} \mathbf{p} r^2 h$ .