

CALCULUS IIE**ASSIGNMENT I****PART 1**

a) Prove using mathematical induction that:

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

For $n = 1$

$$1^2 = \frac{1}{6}(1+1)(2+1)$$

$$1 = \frac{1}{6} \times 2 \times 3$$

$$1 = 1$$

For $n = n + 1$

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 + (n+1)^2 = \frac{1}{6}n(n+1)(n+2)(2n+3)$$

$$LHS = \frac{1}{6}n(n+1)(2n+1) + (n+1)^2$$

$$\frac{1}{6}(n+1)[n(2n+1) + 6(n+1)]$$

$$\frac{1}{6}(n+1)(2n^2 + n + 6n + 6)$$

$$\frac{1}{6}(n+1)(2n^2 + 7n + 6)$$

$$\frac{1}{6}(n+1)(n+2)(2n+3) = RHS \quad QED$$

Use the formula above to:

(i) Find the value of $1^2 + 2^2 + 3^2 + \dots + 10^2$

$$\sum_{i=1}^{10} i^2 = 1^2 + 2^2 + 3^2 + \dots + 10^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\frac{1}{6} \times 10(10+1)(20+1)$$

$$\frac{1}{6} \times 10 \times 11 \times 21$$

= 385

(ii) Find an expression for $\left(\frac{a}{4}\right)^2 + \left(\frac{2a}{4}\right)^2 + \left(\frac{3a}{4}\right)^2 + \dots + \left(\frac{ra}{4}\right)^2$

$$\sum_{i=1}^r \left[i \left(\frac{a}{4}\right) \right]^2 = \left(\frac{a}{4}\right)^2 + \left(\frac{2a}{4}\right)^2 + \left(\frac{3a}{4}\right)^2 + \dots + \left(\frac{ra}{4}\right)^2 = \left(\frac{a}{4}\right)^2 \left[\frac{1}{6} r(r+1)(2r+1) \right]$$

b) Develop a formula for the following

$$\sum_{i=1}^n (2i-1)^3 = 1^3 + 3^3 + 5^3 + \dots + (2n-1)^3$$

n	1	2	3	4	5	6	7	8	9	10
	1 ³	3 ³	5 ³	7 ³	9 ³	11 ³	13 ³	15 ³	17 ³	19 ³
	1	28	153	496	1225	2556	4753	8128	13041	19900
	1 ² × 1	2 ² × 7	3 ² × 17	4 ² × 31	5 ² × 49	6 ² × 71				

= n²(2n² - 1)

Proof by mathematical induction

$$\sum_{i=1}^n (2i-1)^3 = 1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = n^2(2n^2 - 1)$$

For n = 1

1³ = 1²(2 × 1² - 1)
 1 = 2 - 1
 1 = 1

For n = n + 1

$$\sum_{i=1}^n (2i-1)^3 = 1^3 + 3^3 + 5^3 + \dots + [2(n+1)-1]^3 = n^2(2n^2 - 1)$$

(2n + 2 - 1)³
 (2n + 1)³

$$\sum_{i=1}^n (2i-1)^3 = 1^3 + 3^3 + 5^3 + \dots + (2n+1)^3 = (n+1)^2 [2(n+1)^2 - 1]$$

(n² + 2n + 1)(2n² + 4n + 2 - 1)
 (n² + 2n + 1)(2n² + 4n + 1)

$$2n^4 + 4n^3 + n^2 + 4n^3 + 8n^2 + 2n + 2n^2 + 4n + 1$$

$$RHS = 2n^4 + 8n^3 + 11n^2 + 6n + 1$$

$$\sum_{i=1}^n (2i-1)^3 = 1^3 + 3^3 + 5^3 + \dots + (2n+1)^3 = 2n^4 + 8n^3 + 11n^2 + 6n + 1$$

$$LHS = n^2(2n^2 - 1) + (2n+1)^3$$

$$2n^4 - n^2 + (2n+1)(2n+1)(2n+1)$$

$$2n^4 - n^2 + (4n^2 + 2n + 2n + 1)(2n+1)$$

$$2n^4 - n^2 + (4n^2 + 4n + 1)(2n+1)$$

$$2n^4 - n^2 + 8n^3 + 4n^2 + 8n^2 + 4n + 1 + 2n$$

$$2n^4 - n^2 + 8n^3 + 12n^2 + 6n + 1$$

$$2n^4 + 8n^3 + 11n^2 + 6n + 1 = RHS \quad \text{QED}$$

PART 2

Cylinders inside cone

$$V_{3 \text{ cylinders}} = \pi \left(\frac{h}{4} \right) \left(\frac{r}{4} \right)^2 + \pi \left(\frac{h}{4} \right) \left(\frac{r}{2} \right)^2 + \pi \left(\frac{h}{4} \right) \left(\frac{3r}{4} \right)^2$$

$$\pi \frac{h}{4} \left[\left(\frac{r}{4} \right)^2 + \left(\frac{r}{2} \right)^2 + \left(\frac{3r}{4} \right)^2 \right]$$

$$\pi \frac{h}{4} \left(\frac{r^2}{16} + \frac{r^2}{4} + \frac{9r^2}{16} \right)$$

$$\pi \frac{h}{4} r^2 \left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16} \right)$$

$$\pi \left(\frac{h}{4} \right) r^2 \times \frac{7}{8}$$

$$\pi \left(\frac{7h}{32} \right) r^2$$

$$\pi \frac{7}{32} hr^2 = \frac{7}{32} \pi hr^2$$

The volume of a cone is $\frac{1}{3} \pi r^2 h$

$$\frac{1}{3} \pi r^2 h > \frac{7}{32} \pi r^2 h$$

Cylinders outside cone

$$\begin{aligned}
 V_{4 \text{ cylinders}} &= \mathbf{p} \left(\frac{h}{4} \right) \left(\frac{r}{4} \right)^2 + \mathbf{p} \left(\frac{h}{4} \right) \left(\frac{r}{2} \right)^2 + \mathbf{p} \left(\frac{h}{4} \right) \left(\frac{3r}{4} \right)^2 + \mathbf{p} \left(\frac{h}{4} \right) r^2 \\
 &= \mathbf{p} \frac{h}{4} \left(\frac{r^2}{16} + \frac{r^2}{4} + \frac{9r^2}{16} + r^2 \right) \\
 &= \mathbf{p} \left(\frac{h}{4} \right) r^2 \left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1 \right) \\
 &= \mathbf{p} \left(\frac{h}{4} \right) r^2 \times 1 \frac{7}{8} \\
 &= \mathbf{p} \left(\frac{h}{4} \right) r^2 \times \frac{15}{8} \\
 &= \mathbf{p} \left(\frac{15h}{32} \right) r^2 \\
 &= \frac{15}{32} \mathbf{p} h r^2
 \end{aligned}$$

The volume of a cone is $\frac{1}{3} \mathbf{p} r^2 h$

$$\frac{1}{3} \mathbf{p} r^2 h < \frac{15}{32} \mathbf{p} h r^2$$

Investigate how accurate these rules are for cylinders of particular sizes

Cylinders inside cones

Cone of height 1 and radius 2

$$V_{\text{approx}} = \frac{7}{32} \mathbf{p} \times 1 \times 2^2$$

$$\frac{7}{32} \mathbf{p}^4$$

$$= 2.7489$$

$$V_{\text{actual}} = \frac{1}{3} \mathbf{p} \times 1 \times 2^2$$

$$\frac{1}{3} \mathbf{p}^4$$

$$= 4.189$$

Cone of height 5 and radius 15

$$V_{\text{approx}} = \frac{7}{32} \mathbf{p} \times 5 \times 15^2$$

$$V_{\text{actual}} = \frac{1}{3} \mathbf{p} \times 5 \times 15^2$$

$$\begin{aligned}\frac{7}{32}\mathbf{p} \times 5 \times 225 \\ = 773.13\end{aligned}$$

$$\begin{aligned}\frac{1}{3}\mathbf{p} \times 5 \times 225 \\ = 1178.1\end{aligned}$$

Cone of height 10 and radius of 20

$$\begin{aligned}V_{approx} &= \frac{7}{32}\mathbf{p} \times 10 \times 20^2 \\ \frac{7}{32}\mathbf{p} \times 10 \times 400 \\ &= 2748.9\end{aligned}$$

$$\begin{aligned}V_{actual} &= \frac{1}{3}\mathbf{p} \times 10 \times 20^2 \\ \frac{1}{3}\mathbf{p} \times 10 \times 400 \\ &= 4188.8\end{aligned}$$

Cylinders outside cones